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Leibniz rules and reality conditions

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Abstract. An analysis is made of reality conditions within the context of non-commutative geometry. We show that if a covariant derivative satisfies a given left Leibniz rule then a right Leibniz rule fixes the reality condition for the covariant derivative itself. We show also that the map which determines the right Leibniz rule must satisfy the braid equation if the extension of the covariant derivative to tensor products is to satisfy the reality condition.

1 Introduction and motivation

In non-commutative geometry (or algebra), reality conditions are not as natural as they can be in the commutative case; the product of two hermitian elements is no longer necessarily hermitian. The product of two hermitian differential forms is also not necessarily hermitian. It is our purpose here to analyze this problem in some detail. If the reality condition is to be extended to a covariant derivative then we shall show that there is a close correspondence between the existence of the star structure on the tensor product and the existence of a left and right Leibniz rule fulfilling an additional constraint which comes from the condition that the star structure be an involution. We shall show also that the matrix which determines the reality condition must satisfy the Yang–Baxter condition if the extension of the covariant derivative to tensor products is to be well defined. This is equivalent to the braid condition for the map σ which determines the right Leibniz rule. This is necessary in discussing the reality of the curvature form. We shall find in fact that the map σ plays a role completely analogous to that of the braiding map in the pioneering work of Woronowicz [1] in defining real calculi on quantum groups.

There is not as yet a completely satisfactory definition of either a linear connection or a metric within the context of non-commutative geometry but there are definitions which seem to work in certain cases. In the present article we chose the definition of a linear connection as a covariant derivative, a definition which is an adaptation [2] of the definition proposed by Koszul [3] and Connes [4] of a general connection. We shall use therefore the expression "connection" and "covariant derivative" synonymously. We refer to a recent book by one of the present authors [5] for a list of some other examples and references to alternative definitions. More details of one alternative version can be found, for example, in the book by Landi [6]. We refer to this book also for an alternative definition [4] of a metric. For a general introduction to more mathematical aspects of the subject we refer to the book by Connes [4]. We find our results first in the context of a particular version of non-commutative geometry which can be considered as a non-commutative extension of the moving-frame formalism of Cartan. This implies that we suppose that the module of 1-forms is free as a right or left module. As a bimodule it will always be projective with one generator, the generalized "Dirac operator". More details can be found elsewhere [2,5]. Then we rederive the results in a more general context, without using the frame formalism. In the second section we review briefly what we mean by the frame formalism and we recall the particular definition of a covariant derivative which we use. In the third section we discuss the reality condition. We describe here the relation between the map which determines the right Leibniz rule and the map which determines the reality condition. The last section contains the formulation of the main results without the frame formalism and a generalization to higher wedge and tensor powers.

2 The frame formalism

The starting point is a non-commutative associative unital algebra $\mathcal{A}$ and over $\mathcal{A}$ a differential calculus [7,8] $\Omega^*(\mathcal{A})$. We recall that there is a canonical way of constructing a complete differential calculus from the left and right module structure of the $\mathcal{A}$-module of 1-forms $\Omega^1(\mathcal{A})$, a construction which yields the largest calculus which is consistent with the relations which determine the algebra [9,10]. In particular this determines the $\mathcal{A}$-bimodule structure of all the $\Omega^k(\mathcal{A})$. We shall use this construction and we shall restrict our attention to the case where the module is free of rank n as a left or right module.

If $\Omega^1(\mathcal{A})$ possesses a special basis θ^a, $1 \leq a \leq n$, which commutes with the elements f of the algebra,

$$[f, \theta^a] = 0, \tag{2.1}$$

then some of the assumptions we make can be more easily formulated. We therefore start with this case. The existence of this basis implies that in the commutative limit the associated manifold must be parallelizable. We shall refer to the θ^a as a "frame" or "Stehbein". The integer n plays the role of "dimension"; it can be greater than the dimension of the limit manifold but in this case the frame will have a singular limit. Calculations will be even simpler if in addition [11,10] the basis $\{\theta^a\}$ is dual to a set of inner derivations $e_a = \mathrm{ad}\lambda_a$ for some $\lambda_a \in \mathcal{A}$. This means that the differential is given by the expression

$$\mathrm{d}f = e_a f \theta^a = [\lambda_a, f]\theta^a. \tag{2.2}$$

One can rewrite this equation as

$$\mathrm{d}f = -[\theta, f], \tag{2.3}$$

if one introduces [4] the "Dirac operator"

$$\theta = -\lambda_a \theta^a. \tag{2.4}$$

The wedge product between frame elements can be written

$$\theta^a \theta^b = P^{ab}{}_{cd} \theta^c \theta^d, \tag{2.5}$$

where we expect the matrix P to go to the antisymmetric projector in the commutative limit. Because of (2.1), the $P^{ab}{}_{cd}$ belong to the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$, in particular will be complex numbers if the latter is trivial. It can be shown [10,12] that the λ_a must satisfy a quadratic consistency condition. In the simplest case the terms of degree 0,1 vanish and the condition reads

$$\lambda_c \lambda_d P^{cd}{}_{ab} = 0. \tag{2.6}$$

We propose [2] as definition of a linear connection a map [3,4,13]

$$\Omega^1(\mathcal{A}) \xrightarrow{\mathrm{D}} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.7}$$

which satisfies both a left Leibniz rule

$$\mathrm{D}(f\xi) = \mathrm{d}f \otimes \xi + f\mathrm{D}\xi \tag{2.8}$$

and a right Leibniz rule

$$\mathrm{D}(\xi f) = \sigma(\xi \otimes \mathrm{d}f) + (\mathrm{D}\xi)f \tag{2.9}$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$. We have here introduced a generalized permutation

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.10}$$

in order to define a right Leibniz rule which is consistent with the left one. It is necessarily [14,5] $\mathcal{A}$-bilinear:

$$\sigma(f\xi \otimes \eta g) = f\sigma(\xi \otimes \eta)g \tag{2.11}$$

for arbitrary $f, g \in \mathcal{A}$. A linear connection is therefore a couple (D, σ). It can be shown [2] that a necessary as well as sufficient condition for torsion to be right linear is that σ satisfy the consistency condition

$$\pi \circ (\sigma + 1) = 0. \tag{2.12}$$

The map (2.7) has a natural extension [3]

$$\Omega^*(\mathcal{A}) \xrightarrow{D} \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{2.13}$$

to the entire tensor algebra given by a graded Leibniz rule.

This general formalism can be applied in particular to differential calculi with a frame. Since $\Omega^1(\mathcal{A})$ is a free module the map σ can be defined by its action on the basis elements:

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd} \theta^c \otimes \theta^d. \tag{2.14}$$

By the sequence of identities

$$\begin{aligned} fS^{ab}{}_{cd}\theta^c \otimes \theta^d &= \sigma(f\theta^a \otimes \theta^b) = \sigma(\theta^a \otimes \theta^b f) \\ &= S^{ab}{}_{cd} f\theta^c \otimes \theta^d \end{aligned} \tag{2.15}$$

we conclude that the coefficients $S^{ab}{}_{cd}$ must lie in $\mathcal{Z}(\mathcal{A})$.

A covariant derivative is completely specified once its action is assigned on the basis elements:

$$\mathrm{D}\theta^a = -\omega^a{}_{bc}\theta^b \otimes \theta^c. \tag{2.16}$$

The coefficients here are elements of the algebra. The extension of (2.16) to an arbitrary element $\xi \in \Omega^1(\mathcal{A})$ is given by the two Leibniz rules (2.8), (2.9). This implies a condition on the coefficients $\omega^a{}_{bc}$, which is easier to express if the basis is a frame. We have then, for $\xi = \xi_a\theta^a = \theta^a\xi_a$,

$$\mathrm{D}\xi = \mathrm{d}\xi_a \otimes \theta^a - \xi_a\omega^a{}_{bc}\theta^b \otimes \theta^c$$

as well as

$$\mathrm{D}\xi = \sigma(\theta^a \otimes \mathrm{d}\xi_a) - \omega^a{}_{bc}\xi_a\theta^b \otimes \theta^c.$$

These two expressions must be equal as a consistency conditions. The torsion 2-form is defined as usual as

$$\Theta^a = \mathrm{d}\theta^a - \pi \circ \mathrm{D}\theta^a. \tag{2.17}$$

In the case considered in formula (2.6) it is easy to verify [2] that

$$\mathrm{D}_{(0)}\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta) \tag{2.18}$$

defines a torsion-free covariant derivative. This will be a good example to use as a test case.

We shall define a metric as a $\mathcal{A}$-bilinear map [14]

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{g} \mathcal{A}.$$

For any basis of 1-forms $\{\theta^a\}$, g is completely determined by the matrix elements

$$g(\theta^a \otimes \theta^b) = g^{ab}, \tag{2.19}$$

which are elements of $\mathcal{A}$. If in particular we choose a frame, then by the sequence of identities

$$fg^{ab} = g(f\theta^a \otimes \theta^b) = g(\theta^a \otimes \theta^b f) = g^{ab} f \qquad (2.20)$$

one concludes that the coefficients g^{ab} must lie in $\mathcal{Z}(\mathcal{A})$. We define the metric to be symmetric if

$$g \circ \sigma \propto g \qquad (2.21)$$

with a proportionality factor equal to 1 if σ is an involution. This is a natural generalization of the situation in ordinary differential geometry where symmetry is respect to the flip which defines the forms. If $g^{ab} = g^{ba}$ then by a linear transformation of the original λ_a one can make g^{ab} the components of the Euclidean (or Minkowski) metric in dimension n. In general it follows from (2.21) that the components of the metric will not be symmetric in the ordinary sense of the word.

We shall say [14] that the covariant derivative (2.16) is compatible with the metric if and only if the condition

$$\omega^a{}_{bc} + \omega_{cd}{}^e S^{ad}{}_{be} = 0 \qquad (2.22)$$

holds. This is a straightforward "twisted" form of the usual condition that $g_{ad}\omega^d{}_{bc}$ be antisymmetric in the two indices a and c which in turn expresses the fact that for fixed b the $\omega^a{}_{bc}$ form a representation of the Lie algebra of the Euclidean group $SO(n)$ (or the Lorentz group). Under the assumption (2.6) the condition that (2.16) be metric compatible can be written [10] as

$$S^{ae}{}_{df} g^{fg} S^{bc}{}_{eg} = g^{ab}\delta^c_d. \qquad (2.23)$$

Introduce the standard notation $\sigma_{12} = \sigma \otimes 1$ and $\sigma_{23} = 1 \otimes \sigma$ to extend to the tensor product of three copies of a module any operator σ defined on respectively the first two or last two copies. Then there is a natural continuation of the map (2.7) to the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ given by the map

$$\mathrm{D}_2(\xi \otimes \eta) = \mathrm{D}\xi \otimes \eta + \sigma_{12}(\xi \otimes \mathrm{D}\eta). \qquad (2.24)$$

For the covariant derivative (2.18) it is immediate to show that

$$\mathrm{D}_{(0)2}(\xi \otimes \eta) = -\theta \otimes \xi \otimes \eta + \sigma_{12}\sigma_{23}(\xi \otimes \eta \otimes \theta). \qquad (2.25)$$

The map $\mathrm{D}_2 \circ D$ has no particularly interesting properties but if one introduces the notation $\pi_{12} = \pi \otimes 1$ then by analogy with the commutative case one can set

$$\mathrm{D}^2 = \pi_{12} \circ \mathrm{D}_2 \circ D \qquad (2.26)$$

and formally define the curvature as the map

$$\mathrm{Curv} :\ \Omega^1(\mathcal{A}) \longrightarrow \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \qquad (2.27)$$

given by $\mathrm{Curv} = \mathrm{D}^2$. This coincides with the composition of the first two maps of the series of (2.13). Because of the condition (2.12) Curv is left linear. It can be written in terms of the frame as

$$\mathrm{Curv}(\theta^a) = -\frac{1}{2} R^a{}_{bcd}\theta^c\theta^d \otimes \theta^b \qquad (2.28)$$

Similarly one can define a Ricci map

$$\mathrm{Ric}(\theta^a) = \frac{1}{2} R^a{}_{bcd}\theta^c g(\theta^d \otimes \theta^b). \qquad (2.29)$$

It is given by

$$\mathrm{Ric}(\theta^a) = R^a{}_b\theta^b. \qquad (2.30)$$

The above definition of curvature is not satisfactory in the non-commutative case [2]. For example, from (2.28) one sees that Curv can only be right linear if $R^a{}_{bcd} \in \mathcal{Z}(\mathcal{A})$.

The curvature $\mathrm{Curv}_{(0)}$ of the covariant derivative $\mathrm{D}_{(0)}$ defined in (2.18) can be readily calculated. One finds after a short calculation that it is given by the expression

$$\mathrm{Curv}_{(0)}(\theta^a) = \theta^2 \otimes \theta^a + \pi_{12}\sigma_{12}\sigma_{23}\sigma_{12}(\theta^a \otimes \theta \otimes \theta). \qquad (2.31)$$

If $\xi = \xi_a\theta^a$ is a general 1-form then since Curv is left linear one can write

$$\mathrm{Curv}_{(0)}(\xi) = \xi_a\theta^2 \otimes \theta^a + \pi_{12}\sigma_{12}\sigma_{23}\sigma_{12}(\xi \otimes \theta \otimes \theta). \qquad (2.32)$$

The lack of right linearity of Curv is particularly evident in this last formula.

3 The involution

Suppose now that $\mathcal{A}$ is a algebra with an hermitian adjoint $f \mapsto f^*$ and with a compatible differential calculus. It means that there is an antilinear map $\jmath_1$ of $\Omega^1(\mathcal{A})$ into itself such that, for any $f, h \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$,

$$\jmath_1(\mathrm{d}f) \equiv (\mathrm{d}f)^* = \mathrm{d}f^* \qquad (3.1)$$

and

$$\jmath_1(f\xi h) \equiv (f\xi h)^* = h^*\xi^* f^*. \qquad (3.2)$$

This amounts to the notion of a $*$-calculus originally introduced by Woronowicz in [1]. Some differential calculi satisfy this condition (see e.g. [1,16,17,11,18]), but not all (see e.g. [19]). If there is a Dirac operator θ then it follows necessarily that it is an antihermitian 1-form:

$$\theta^* = -\theta. \qquad (3.3)$$

If a frame exists, ons can apply the involution to (2.1). Using the relation (3.2) we conclude that there must exist coefficients $C^a_b \in \mathcal{Z}(\mathcal{A})$ such that

$$(\theta^a)^* = C^a_b\theta^b. \qquad (3.4)$$

Since the adjoint is an involution then necessarily $(C^a_c)^* C^c_b = \delta^a_b$. This is not to be confused with the condition $g_{cd}(C^c_a)^* C^d_b = g_{ab}$ that C be unitary. This is why one cannot always transform an arbitrary frame into a real frame. We shall be mainly concerned with the case in which one can impose the condition

$$(\theta^a)^* = \theta^a. \qquad (3.5)$$

Equations (2.2), (3.1) and (3.5) imply then that for the dual inner derivations

$$(e_a f^*)^* = e_a f, \qquad (3.6)$$

which in turn implies that the λ_a are antihermitian. Conversely of course, if the λ_a are chosen antihermitian then the condition (3.5) is necessarily satisfied.

We shall extend the involution to a map

$$\jmath_n : \bigotimes_1^n \Omega^1(\mathcal{A}) \to \bigotimes_1^n \Omega^1(\mathcal{A}), \tag{3.7}$$

of tensor powers of $\Omega^1(\mathcal{A})$ into themselves for $n > 1$. Consider first the case $n = 2$. This map would in turn allow one to extend the involution $\imath_1 \equiv \jmath_1$ of $\Omega^1(\mathcal{A})$ to an involution $\imath_2$ of $\Omega^2(\mathcal{A})$ by requiring that it be compatible with the action of the product:

$$\imath_2 \circ \pi = \pi \circ \jmath_2. \tag{3.8}$$

Our main requirement is the reality condition

$$\mathrm{D}\xi^* = (\mathrm{D}\xi)^* \tag{3.9}$$

on the connection and on $\jmath_2$, a condition which can be rewritten also in the form

$$\mathrm{D} \circ \jmath_1 = \jmath_2 \circ \mathrm{D}. \tag{3.10}$$

This must be consistent with the Leibniz rules (2.8) and (2.9). From the equalities

$$(\mathrm{D}(f\xi))^* = \mathrm{D}((f\xi)^*) = \mathrm{D}(\xi^* f^*) \tag{3.11}$$

one finds the condition

$$(\mathrm{d}f \otimes \xi)^* + (f\mathrm{D}\xi)^* = \sigma(\xi^* \otimes \mathrm{d}f^*) + (\mathrm{D}\xi)^* f^*. \tag{3.12}$$

The latter will be satisfied if we define the involution in general by (see also [14,20])

$$(\xi \otimes \eta)^* = \sigma(\eta^* \otimes \xi^*). \tag{3.13}$$

It follows immediately that the first terms on either side of (3.12) are equal, whereas the equality of the second terms is a direct consequence of (3.13) and of the $\mathcal{A}$-bilinearity of σ, that for any $f, h \in \mathcal{A}$ and $\xi, \eta \in \Omega^1(\mathcal{A})$

$$(f\xi \otimes \eta h)^* = h^*(\xi \otimes \eta)^* f^*. \tag{3.14}$$

From (3.13) we see that a change in σ implies a change in the definition of an hermitian tensor. Thus, there is an intimate connection between the reality condition and the right Leibniz rule and it follows that $\jmath_2$ is also restricted by the condition (2.12). Equation (3.13) can be also read from right to left as a definition of the right Leibniz rule in terms of the hermitian structure. Note also that the involution (3.13) has the ordinary flip as a commutative limit, since in this limit σ become the ordinary flip. (Therefore it is related to, but should not be confused with, the particular involution on $\Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$ introduced by Woronowicz [1] in the proof of his Theorem 3.4.)

If in (3.13) we choose $\xi = \theta^a$ and $\eta = \theta^b$, then we find that

$$(\theta^a \otimes \theta^b)^* \equiv \jmath_2(\theta^a \otimes \theta^b) = J^{ab}{}_{cd}\theta^c \otimes \theta^d, \tag{3.15}$$

where the coefficients $J^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A})$ are given by

$$J^{ab}{}_{cd} = C^b_e C^a_f S^{ef}{}_{cd}. \tag{3.16}$$

If the frame is real this becomes the equality

$$J^{ab}{}_{cd} = S^{ba}{}_{cd}. \tag{3.17}$$

The condition that the star operation be an involution places a further constraint on the map σ:

$$(\sigma(\eta^* \otimes \xi^*))^* = (\xi \otimes \eta). \tag{3.18}$$

In terms of the frame this gives the condition

$$(J^{ab}{}_{cd})^* J^{cd}{}_{ef} = \delta^a_e \delta^b_f, \tag{3.19}$$

which for a real frame yields in turn the condition

$$(S^{ba}{}_{dc})^* = (S^{-1})^{ab}{}_{cd}. \tag{3.20}$$

What we have required so far was necessary, but not sufficient to make the connection real. From (2.16) written in terms of the frame the reality condition (3.10) to be fulfilled now reduces to the constraint

$$C^a_d \omega^d{}_{bc} = (\omega^a{}_{de})^* J^{de}{}_{bc}. \tag{3.21}$$

In particular, for a real frame this is equivalent to the constraint

$$\omega^a{}_{bc}(S^{cb}{}_{de})^* = (\omega^a{}_{de})^*. \tag{3.22}$$

Using (3.13), (3.18) and (3.3) one verifies immediately that the connection (2.18) is real.

We would like now to use $\jmath_2$ to define the involution on the algebra of 2-forms through the standard [1] projection procedure (3.8). Using (2.12), we find that for any $\xi, \eta \in \Omega^1(\mathcal{A})$ we must have

$$(\xi\eta)^* = \pi \circ \sigma(\eta^* \otimes \xi^*) = -\eta^*\xi^*. \tag{3.23}$$

For an hermitian calculus this is automatically consistent with the product since this as well as the structure of the algebra $\Omega^*(\mathcal{A})$ is fully encoded in the module structure of $\Omega^1(\mathcal{A})$. In terms of the frame this implies that the commutation relations (2.5)

$$\theta^a\theta^b = P^{ab}{}_{cd}\theta^c\theta^d$$

are consistent with (3.23). By applying $\imath_2$ to both sides one can check that this is equivalent to the condition

$$P^{ab}{}_{cd} = (C^a_e)^*(C^b_f)^*(P^{fe}{}_{gh})^* C^g_c C^h_d, \tag{3.24}$$

which for a real frame simplifies to

$$P^{ab}{}_{cd} = (P^{ba}{}_{dc})^*. \tag{3.25}$$

From (3.23) it follows that the exterior derivative is real also on 1-forms. This is immediate if d can be realized as an anticommutator with θ, because of (3.3). More generally it follows from the sequence of identities

$$\begin{aligned}(\mathrm{d}(f\mathrm{d}g))^* &= (\mathrm{d}f\mathrm{d}g)^* = -(\mathrm{d}g)^*(\mathrm{d}f)^* = -\mathrm{d}g^*\mathrm{d}f^* \\ &= \mathrm{d}((\mathrm{d}g^*)f^*) = \mathrm{d}(f\mathrm{d}g)^*.\end{aligned} \tag{3.26}$$

We shall say that the metric is real if its action commutes with the corresponding involution:

$$g((\xi \otimes \eta)^*) = (g(\xi \otimes \eta))^*. \tag{3.27}$$

In terms of the frame this condition puts the further constraints

$$S^{ab}{}_{cd} g^{cd} = (g^{ba})^* \tag{3.28}$$

on the matrix of coefficients g^{ab} and on $S^{ab}{}_{cd}$.

We shall now consider third tensor powers of $\Omega^1(\mathcal{A})$. In order for the linear curvature associated to D to be real we must require that the extension of the involution to the tensor product of three elements of $\Omega^1(\mathcal{A})$ be such that

$$\pi_{12} \circ \mathrm{D}_2(\xi \otimes \eta)^* = (\pi_{12} \circ \mathrm{D}_2(\xi \otimes \eta))^* . \tag{3.29}$$

We shall impose a stronger condition. We shall require that D_2 be real:

$$\mathrm{D}_2(\xi \otimes \eta)^* = (\mathrm{D}_2(\xi \otimes \eta))^*. \tag{3.30}$$

This can be rewritten also in the form

$$\mathrm{D}_2 \circ \jmath_2 = \jmath_3 \circ \mathrm{D}_2. \tag{3.31}$$

Again, this must be consistent with the Leibniz rules (2.8) and (2.9). Replacing ξ by $f\xi$ in (3.30) and using (2.24) and (3.13) we find that

$$\begin{aligned}&\sigma_{12}\sigma_{23}\sigma_{12}(\eta^* \otimes \xi^* \otimes \mathrm{d}f^*) + (\mathrm{D}_2(\xi \otimes \eta))^* f^* \\ &= (\mathrm{d}f \otimes \xi \otimes \eta)^* + (f\mathrm{D}_2(\xi \otimes \eta))^* .\end{aligned} \tag{3.32}$$

In order to fulfill this condition we are led to the equation

$$(\xi\otimes\eta\otimes\zeta)^* \equiv \jmath_3(\xi\otimes\eta\otimes\zeta) = \sigma_{12}\sigma_{23}\sigma_{12}(\zeta^*\otimes\eta^*\otimes\xi^*). \tag{3.33}$$

The second terms on either side of (3.32) are manifestly equal, whereas the first terms are equal as a direct consequence of (3.33). The condition that the map $\jmath_3$ be an involution places a further constraint on the map σ. In order to make this explicit we first assume for simplicity that there exists a real frame θ^a. It follows then that

$$\begin{aligned}(\theta^a \otimes \theta^b \otimes \theta^c)^* &= \sigma_{12}\sigma_{23}\sigma_{12}(\theta^c \otimes \theta^b \otimes \theta^b) \\ &= (S_{12}S_{23}S_{12})^{cba}{}_{def}\theta^d \otimes \theta^e \otimes \theta^f,\end{aligned} \tag{3.34}$$

where

$$(S_{12})^{abc}{}_{def} = S^{ab}{}_{de}\delta^c_f, \quad (S_{23})^{abc}{}_{def} = \delta^a_d S^{bc}{}_{ef},$$

and the constraint reads

$$\left((S_{12}S_{23}S_{12})^{cba}{}_{def}\right)^* (S_{12}S_{23}S_{12})^{fed}{}_{pqr} = \delta^a_p\delta^b_q\delta^c_r. \tag{3.35}$$

Using (3.20) it is easy to check that this is equivalent to the braid equations

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \tag{3.36}$$

for the matrix S and

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23} \tag{3.37}$$

for the map σ. We are thus led to postulate that σ satisfies also (3.37). As we shall show in next section, this will make $\jmath_3$ an involution in the general case, even if there is no real frame. In terms of the matrix $J^{ab}{}_{cd}$ (3.36) becomes the Yang–Baxter equation

$$J_{12}J_{13}J_{23} = J_{23}J_{13}J_{12}, \tag{3.38}$$

which is the correct expression of (3.37) in the frame formalism also if the frame is not real.

Having defined a consistent involution $\jmath_3$ by (3.33), we now look for sufficient conditions for D_2 to fulfill (3.30). Again, we first assume for simplicity that there exists a real frame θ^a. In terms of the frame, from the definition (2.24) of D_2 one has

$$\mathrm{D}_2(\theta^a \otimes \theta^b) = -(\omega^a{}_{pq}\delta^b_r + S^{ac}{}_{pq}\omega^b{}_{cr})\theta^p \otimes \theta^q \otimes \theta^r, \tag{3.39}$$

and the condition (3.30) becomes

$$\begin{aligned}&S^{ba}{}_{pq}(\omega^p{}_{de}\delta^q_f + S^{pr}{}_{de}\omega^q{}_{rf}) = \\ &\left((\omega^a{}_{pq})^*\delta^b_r + (S^{as}{}_{pq})^*(\omega^b{}_{sr})^*\right)(S_{12}S_{23}S_{12})^{rqp}{}_{def}.\end{aligned} \tag{3.40}$$

Using (3.22), (3.36) and (3.20) it is easy to verify that the latter is equivalent to the equation

$$S^{ba}{}_{pe}\omega^p{}_{cd} + S^{ba}{}_{pq}S^{pr}{}_{cd}\omega^q{}_{re} = S^{pa}{}_{de}\omega^b{}_{cp} + S^{bq}{}_{cp}S^{pr}{}_{de}\omega^a{}_{qr}. \tag{3.41}$$

The latter can be rewritten more abstractly and concisely in the form

$$\mathrm{D}_2 \circ \sigma = \sigma_{23} \circ \mathrm{D}_2. \tag{3.42}$$

In fact, the most general element in $\Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$ can be written in the form $f_{ab}\theta^a \otimes \theta^b$. Applying (3.42) to this expression and using (3.39) and (2.8) one finds

$$\begin{aligned}&\mathrm{D}_2\left(\sigma(f_{ab}\theta^b \otimes \theta^a)\right) - \sigma_{23} \circ \left(\mathrm{D}_2(f_{ab}\theta^b \otimes \theta^a)\right) \\ &= S^{ba}{}_{pq}\mathrm{D}_2(f_{ab}\theta^p \otimes \theta^q) \\ &-\mathrm{d}f_{ab} \otimes \sigma(\theta^b \otimes \theta^a) \\ &-f_{ab}(\omega^b{}_{cp}\delta^a_r + S^{bq}{}_{cp}\omega^a{}_{qr})\sigma_{23}(\theta^c \otimes \theta^p \otimes \theta^r) \\ &= f_{ab}\left(S^{ba}{}_{pq}(\omega^p{}_{cd}\delta^q_e + S^{pr}{}_{cd}\omega^q{}_{re})\right. \\ &\left.-(\omega^b{}_{cp}\delta^a_r + S^{bq}{}_{cp}\omega^a{}_{qr})S^{pr}{}_{de}\right)\theta^c \otimes \theta^d \otimes \theta^e.\end{aligned}$$

The right-hand side of this equation vanishes if and only if (3.41) holds. To summarize, then, besides (3.37) and (3.18) we are led to postulate that σ fulfills also (3.42). As we shall show in next section, this will make D_2 real in the general case, even if there is no real frame. Using (2.25) and (3.37) it is easy to check that the covariant derivative (2.18) satisfies (3.42).

To define the involution on 3-forms we apply $\pi_{23} \circ \pi_{12}$ to (3.33). Using repeatedly (2.12) and the associativity $\pi_{23} \circ \pi_{12} = \pi_{12} \circ \pi_{23}$ of the wedge product we find that

$$\begin{aligned}(\xi\eta\zeta)^* &\equiv \pi_{23} \circ \pi_{12} \circ \jmath_3(\xi \otimes \eta \otimes \zeta) = -\zeta^*\eta^*\xi^* \\ &= \zeta^*(\xi\eta)^* = (\eta\zeta)^*\xi^*.\end{aligned} \tag{3.43}$$

Moreover, reasoning as before, one finds that the exterior derivative is real also on 2-forms. Note that our construction is based on the existence of a $\mathcal{A}$-bilinear map σ fulfilling the braid condition (3.37) and the constraint (3.18).

It is remarkable that this is exactly what was used by Woronowicz [1] to define real calculi on quantum groups, although we are here in a rather different context. Because of (3.37), σ acts like a braiding in Hopf algebra theory.

We conclude this section by considering some examples where the above construction applies. If in particular the frame is real and $P^{ab}{}_{cd}$ is given by

$$P^{ab}{}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) \tag{3.44}$$

we can choose σ to be the ordinary flip and $\jmath_2$ the identity. As explicit examples where P^{ab}_{cd} are of the form (3.44), and the differential calculi are based on real derivations, we mention the calculi based on lie algebras of derivations of matrix algebras [11,18] and the Jordanian deformation [17,21] of the plane.

On real manifolds one considers usually only real derivations and real forms but it is however often of interest on even-dimensional manifolds to introduce derivations which are not real and use them to define an involution on the module of 1-forms known as an almost-complex structure. We have given the general expression which the extension of the involution must have to $\Omega^2(\mathcal{A})$ as well as to $\Omega^1(\mathcal{A})\otimes\Omega^1(\mathcal{A})$; this again parallels the procedure which is used in ordinary geometry. The involution, we have seen, depends on the form of the projector π, notably on its coefficients $P^{ab}{}_{cd}$. The calculi on the quantum Euclidean spaces [22] can be shown [23] to be based on derivations which do not satisfy the condition (3.6). There is in these cases no satisfactory involution on the algebra of forms, at least none which respects the action of the respective quantum groups.

Reality conditions can be introduced also on calculi which neither are based on derivations nor have a frame. We would like to mention also a very simple example, the calculus known as the Connes–Lott model [24] which is based on the algebra of 3×3 matrices. The 1-forms in this example can be identified as matrices and the involution is defined to be the ordinary involution on matrices. This model is of interest not only because it furnishes an example of an involution on a calculus which is not based on derivations but because it is the simplest example of a general procedure [4,25] which allows one to introduce involutions on differential calculi over algebras of operators, a procedure which is based on an operator used in the theory of von Neumann algebras known as the modular conjugation operator and which is generally denoted also by J. There have been several [26,27] recent expositions of this operator within the present context. The discussion we have given is in principle valid for arbitrary associative algebras, not necessarily operator algebras, but if one is considering an algebra which is represented as an operator algebra and with a differential calculus defined by a generalized Dirac operator [4,25], and with a frame as we have defined it then the involution induced by the modular conjugation operator on the frame would satisfy all of the conditions we have described for the involution $\jmath$ on 1-forms and each representation would define a different, in general inequivalent, one. We recall that there is in general on a manifold no unique way of defining an almost-complex structure. We cannot discuss this in further detail since no concrete examples are known except for the trivial matrix algebras.

4 Higher tensor and wedge powers

In the present section we rederive the main results of the previous section and generalize them to higher tensor and wedge powers. Apart from the basic assumptions (3.1) and (3.2) that we are using a real calculus, and (3.10) and (3.18) that the covariant derivative of a 1-form be consistent with the involution, we shall use the braid condition (3.37) as well as a consistency condition (3.42) on the extension of the covariant derivative to tensor products. We do not assume the existence of a frame.

Just as we have in (2.24) defined D_2 we can introduce a set D_n of covariant derivatives

$$\mathrm{D}_n : \bigotimes_1^n \Omega^1(\mathcal{A}) \longrightarrow \bigotimes_1^{n+1} \Omega^1(\mathcal{A}) \tag{4.1}$$

for arbitrary integer n by using σ to place the operator D in its natural position to the left. For instance,

$$\mathrm{D}_3 = (\mathrm{D}\otimes 1\otimes 1 + \sigma_{12}(1\otimes\mathrm{D}\otimes 1) + \sigma_{12}\sigma_{23}(1\otimes 1\otimes D)) \tag{4.2}$$

These D_n will also be real in the sense that

$$\mathrm{D}_n\circ\jmath_n = \jmath_{n+1}\circ\mathrm{D}_n \tag{4.3}$$

where the $\jmath_n$ are the natural generalizations of $\jmath_2$ and $\jmath_3$. For instance, $\jmath_4$ is defined by

$$\begin{aligned}(\xi\otimes\eta\otimes\zeta\otimes\omega)^* &\equiv \jmath_4(\xi\otimes\eta\otimes\zeta\otimes\omega)\\ &= \sigma_{12}\sigma_{23}\sigma_{12}\sigma_{34}\sigma_{23}\sigma_{12}(\omega^*\otimes\zeta^*\otimes\eta^*\otimes\xi^*).\end{aligned} \tag{4.4}$$

The general rule to construct $\jmath_n$ is the following. Let ϵ denote the "flip" or the "permutator" of two objects, $\epsilon(\xi\otimes\eta) = \eta\otimes\xi$, and more generally let ϵ_n denote the inverse-order permutator of n objects. For instance, the action of ϵ_3 is given by

$$\epsilon_3(\zeta\otimes\eta\otimes\xi) = \xi\otimes\eta\otimes\zeta. \tag{4.5}$$

The maps ϵ,ϵ_n are $\mathbf{C}$-bilinear but not $\mathcal{A}$-bilinear, and are involutive. One can decompose ϵ_n as a product of $\epsilon_{i(i+1)}$. One finds for $n=3$

$$\epsilon_3 = \epsilon_{12}\epsilon_{23}\epsilon_{12} = \epsilon_{23}\epsilon_{12}\epsilon_{23}. \tag{4.6}$$

The second equality expresses the fact that ϵ fulfills the braid equation. In a more abstract but compact notation the definitions (3.13), (3.33) and (4.4) can be written in the form

$$\jmath_2 = \sigma\ell_2, \tag{4.7}$$

$$\jmath_3 = \sigma_{12}\sigma_{23}\sigma_{12}\ell_3, \tag{4.8}$$

$$\jmath_4 = \sigma_{12}\sigma_{23}\sigma_{12}\sigma_{34}\sigma_{23}\sigma_{12}\ell_4. \tag{4.9}$$

We have here defined the involution on the 1-forms as $\jmath_1$, and

$$\ell_n = (\underbrace{\jmath_1 \otimes \ldots \otimes \jmath_1}_{n \text{ times}})\epsilon_n. \tag{4.10}$$

The ℓ_n is clearly an involution, since ϵ_n commutes with the tensor product of the $\jmath_1$'s. The products of σ's appearing in the definitions of $\jmath_3, \jmath_4$ are obtained from the decompositions of ϵ_3, ϵ_4 by replacing each $\epsilon_{i(i+1)}$ by $\sigma_{i(i+1)}$. In this way, $\jmath_3, \jmath_4$ have the correct classical limit, since in this limit σ become the ordinary flip ϵ. In the same way as different equivalent decompositions of ϵ_3, ϵ_4 are possible, so different products of σ factors in (4.8) and (4.9) are allowed. However there is no ambiguity because they are all equal, once (3.37) is fulfilled. The same rules described for $n = 3, 4$ should be used to define $\jmath_n$ for $n > 4$.

The definition of $\jmath_n$ can be given also some equivalent recursive form which will be useful for the proofs below, namely

$$\begin{aligned} \jmath_3 &= \sigma_{12}\sigma_{23}\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_2), && (4.11)\\ \jmath_4 &= \sigma_{12}\sigma_{23}\sigma_{34}\epsilon_{34}\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_3), && (4.12)\\ &= \sigma_{23}\sigma_{34}\sigma_{12}\sigma_{23}\epsilon_{23}\epsilon_{12}\epsilon_{34}\epsilon_{23}(\jmath_2 \otimes \jmath_2), && (4.13) \end{aligned}$$

and so forth to higher orders. Again, these definitions are unambiguous because of the braid equation (3.37).

Now we wish to show that, if the braid equation is fulfilled and $\jmath_2$ is an involution, that is, (3.18) is satisfied, then $\jmath_n$ is also an involution for $n > 2$. Note that the constraint (3.18) in the more abstract notation introduced above becomes

$$\jmath_2 = \jmath_2^{-1} = \epsilon \circ (\jmath_1 \otimes \jmath_1) \circ \sigma^{-1}. \tag{4.14}$$

As a first step one checks that for $i = 1, \cdots, n-1$

$$\sigma_{i(i+1)}\ell_n = \ell_n\, \sigma^{-1}_{(n-i)(n+1-i)}. \tag{4.15}$$

The latter relation can be proved recursively. We show in particular how from the relation with $n = 2$ follows the relation with $n = 3$:

$$\begin{aligned} \sigma_{12}\ell_3 &\overset{(4.5)}{=} \sigma_{12}(\jmath_1 \otimes \jmath_1 \otimes \jmath_1)\epsilon_{12}\epsilon_{23}\epsilon_{12}\sigma_{23}\sigma_{23}^{-1}\\ &\overset{(4.7)}{=} (\jmath_2 \otimes \jmath_1)\epsilon_{23}\epsilon_{12}\sigma_{23}\sigma_{23}^{-1}\\ &= (\jmath_2\sigma \otimes \jmath_1)\epsilon_{23}\epsilon_{12}\sigma_{23}^{-1}\\ &\overset{(4.14)}{=} (\jmath_1 \otimes \jmath_1 \otimes \jmath_1)\epsilon_{12}\epsilon_{23}\epsilon_{12}\sigma_{23}^{-1}\\ &\overset{(4.5)}{=} \ell_3\sigma_{23}^{-1}. \end{aligned} \tag{4.16}$$

Now it is immediate to show that $\jmath_n$ is an involution. Again, we explicitly reconsider the case $n = 3$:

$$\begin{aligned} (\jmath_3)^2 &= \sigma_{12}\sigma_{23}\sigma_{12}\ell_3\sigma_{23}\sigma_{12}\sigma_{23}\ell_3\\ &\overset{(4.16)}{=} \ell_3\sigma_{23}^{-1}\sigma_{12}^{-1}\sigma_{23}^{-1}\sigma_{23}\sigma_{12}\sigma_{23}\ell_3\\ &= 1. \end{aligned}$$

In order to prove (4.3) it is useful to prove first a direct consequence of relation (3.42):

$$\mathrm{D}_n \circ \sigma_{(i-1)i} = \sigma_{i(i+1)} \circ \mathrm{D}_n. \tag{4.17}$$

The recursive proof is straightforward. For instance,

$$\begin{aligned} \mathrm{D}_3\sigma_{23} &= [\mathrm{D} \otimes 1 \otimes 1 + \sigma_{12}(1 \otimes \mathrm{D}_2)]\sigma_{23}\\ &\overset{(3.42)}{=} \sigma_{34}(\mathrm{D} \otimes 1 \otimes 1) + \sigma_{12}\sigma_{34}(1 \otimes \mathrm{D}_2)\\ &= \sigma_{34}\mathrm{D}_3. \end{aligned}$$

Now (4.3) can be proved recursively. For instance,

$$\begin{aligned} \mathrm{D}_3\jmath_3 &\overset{(4.11)}{=} \mathrm{D}_3\sigma_{12}\sigma_{23}\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_2)\\ &\overset{(4.17)}{=} \sigma_{23}\sigma_{34}\mathrm{D}_3\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_2)\\ &\overset{(4.2)}{=} \sigma_{23}\sigma_{34}[\mathrm{D}_2 \otimes 1\\ &+ \sigma_{12}\sigma_{23}(1 \otimes 1 \otimes \mathrm{D})]\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_2)\\ &= \sigma_{23}\sigma_{34}[\epsilon_{34}\epsilon_{23}\epsilon_{12}(1 \otimes \mathrm{D}_2)\\ &+ \sigma_{12}\sigma_{23}\epsilon_{23}\epsilon_{12}\epsilon_{34}\epsilon_{23}(\mathrm{D} \otimes 1 \otimes 1)](\jmath_1 \otimes \jmath_2)\\ &\overset{(3.31)}{=} \sigma_{23}\sigma_{34}[\epsilon_{34}\epsilon_{23}\epsilon_{12}(\jmath_1 \otimes \jmath_3\mathrm{D}_2)\\ &+ \sigma_{12}\sigma_{23}\epsilon_{23}\epsilon_{12}\epsilon_{34}\epsilon_{23}(\jmath_2\mathrm{D} \otimes \jmath_2)]\\ &\overset{(4.12),(4.13)}{=} \sigma_{12}^{-1}\jmath_4(1 \otimes \mathrm{D}_2) + \jmath_4(\mathrm{D} \otimes 1 \otimes 1)\\ &= \jmath_4[\sigma_{12}(1 \otimes \mathrm{D}_2) + (\mathrm{D} \otimes 1 \otimes 1)]\\ &\overset{(4.2)}{=} \jmath_4\mathrm{D}_3. \end{aligned} \tag{4.18}$$

For the second-last equality we have used the relation $\sigma_{12}^{-1}\jmath_4 = \jmath_4\sigma_{12}$, which can be easily proven using (3.37) and (4.15).

The map $\jmath_k$ allows one to extend the involution also to k-forms by requiring its compatibility with the action of k-form projectors:

$$\pi_{12}\circ\pi_{23}\circ\cdots\circ\pi_{(k-1)k}\circ\jmath_k = *\circ\pi_{12}\circ\pi_{23}\circ\cdots\circ\pi_{(k-1)k}. \tag{4.19}$$

For arbitrary $\alpha_p \in \Omega^p(\mathcal{A})$, $\alpha_q \in \Omega^q(\mathcal{A})$ one finds the general rule

$$(\alpha_p\alpha_q)^* = (-)^{pq}\alpha_q^*\alpha_p^* \tag{4.20}$$

which generalizes (3.23) and (3.43). Reasoning as in the previous section one finds that d is real on all of $\Omega^*(\mathcal{A})$.

For further developments it is convenient to extend the braiding σ functorially to tensor powers of $\Omega^1(\mathcal{A})$ or of $\Omega^*(\mathcal{A})$ (for an introduction to braidings we refer to Majid [28,29]). This is possible because of (3.37). In this framework, the bilinear map σ can be naturally extended first to higher tensor powers of $\Omega^1(\mathcal{A})$,

$$\sigma : (\underbrace{\Omega^1 \otimes \ldots \otimes \Omega^1}_{p \text{ times}}) \otimes (\underbrace{\Omega^1 \otimes \ldots \otimes \Omega^1}_{k \text{ times}}) \to \underbrace{\Omega^1 \otimes \ldots \otimes \Omega^1}_{p+k \text{ times}}. \tag{4.21}$$

This extension can be found by iteratively applying the rules

$$\begin{aligned} \sigma\left((\xi \otimes \eta) \otimes \zeta\right) &= \sigma_{12}\sigma_{23}(\xi \otimes \eta \otimes \zeta),\\ \sigma\left(\xi \otimes (\eta \otimes \zeta)\right) &= \sigma_{23}\sigma_{12}(\xi \otimes \eta \otimes \zeta). \end{aligned} \tag{4.22}$$

Here ξ, η, ζ are elements of three arbitrary tensor powers of $\Omega^1(\mathcal{A})$. It is easy to show that there is no ambiguity in the iterated definitions, and that the extended map still satisfies the braid equation (3.37). These are general properties of a braiding.

Thereafter, by applying $p+k-2$ times the projector π to the previous equation, so as to transform the relevant tensor products into wedge products, σ can be extended also as a map

$$\sigma : \Omega^p(\mathcal{A}) \otimes \Omega^k(\mathcal{A}) \to \Omega^k(\mathcal{A}) \otimes \Omega^p(\mathcal{A}). \tag{4.23}$$

For instance, we shall define σ on $\Omega^2 \otimes \Omega^1$ and $\Omega^1 \otimes \Omega^2$ respectively through

$$\begin{aligned} \sigma(\xi\eta \otimes \zeta) &= \pi_{23}\sigma\left((\xi \otimes \eta) \otimes \zeta\right), \\ \sigma(\xi \otimes \eta\zeta) &= \pi_{12}\sigma\left(\xi \otimes (\eta \otimes \zeta)\right). \end{aligned} \tag{4.24}$$

Under suitable assumptions on π, the extended σ still satisfies the braid equation (3.37). It follows that the same formulae presented above in this section can be used to extend the involutions $\jmath_n$ to tensor powers of higher degree forms in a compatible way with the action of π, that is, in such a way that $\jmath_2 \circ \pi_{12} = \pi_{12} \circ \jmath_3$, and so forth. Finally, also the covariant derivatives D_n can be extended to tensor powers of higher degree forms in such a way that (4.3) is still satisfied. These results will be shown in detail elsewhere.

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